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On the Variation of Uncanonical Arbitrary Constants; with an application to the Planetary Theory.

By A. Y. G. Campbell, B.A., Scholar of Trinity College, Cambridge.

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In attacking the planetary or lunar theories by the method of the variation of arbitrary constants, it is necessary to evaluate several functions usually denoted by $[a, e]$, &c., unless a canonical set of arbitrary constants is chosen. In this paper I endeavour to show that the equations giving the variations of these constants can be found more quickly without evaluating the functions $[a, e]$, &c. It is necessary to find one function (denoted in this paper by C with a suffix) corresponding to each arbitrary constant. In the first two sections the theory and its application to the elliptic constants of the planetary theory are briefly explained. In the following section it is shown that if any solution of the new equations is found we can proceed to find the equations, giving the variation of the new arbitrary constants. The next approximation can be obtained in a manner similar to that used in the previous case. It is shown also in the last section that in certain cases there exists a method of solution similar to Jacobi's method of solution of the Hamiltonian canonical equations. Among these cases is included the set of equations giving the variations of the elliptic elements in the planetary theory.

§ 1.

Let $\alpha_1, \alpha_2, \ldots \alpha_{2n}$ be the set of arbitrary constants of any solution of the canonical equations

$$\dot{q}_i = \frac{dH}{dp_i}, \quad -\dot{p}_i = \frac{dH}{dq_i} (i = 1, 2, \ldots n) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It is required to find the equations giving the rates of variation of $\alpha_1, \alpha_2, \ldots \alpha_{2n}$ when a disturbing function Ω is added to H.

If L is the Lagrangian function corresponding to H,

$$\frac{dL}{d\dot{q}_i} = p_i, \quad \frac{dL}{dq_i} = \dot{p}_i (i = 1, 2, \ldots n);$$

and if α_r is one of the arbitrary constants in terms of which L is supposed to be expressed,

$$\begin{aligned}\frac{dL}{d\alpha_r} &= \sum_{i=1}^{i=n}\left(\frac{dL}{dq_i}\frac{dq_i}{d\alpha_r} + \frac{dL}{d\dot{q}_i}\frac{d\dot{q}_i}{d\alpha_r}\right)\\ &= \sum_{i=1}^{i=n}\left(\dot{p}_i\frac{dq_i}{d\alpha_r} + p_i\frac{d\dot{q}_i}{d\alpha}\right)\\ &= \frac{d}{dt}\left(\sum_{i=1}^{i=n} p_i\frac{dq_i}{d\alpha_r}\right)\end{aligned}$$

the terms in the bracket being expressed in terms of the arbitrary constants, &c. In the above $\dot{p}_i$, $\dot{q}_i$ have been written for the values of

$$\frac{dp_i}{dt},\ \frac{dq_i}{dt},$$

when p_i, q_i are expressed in terms of $a_1, a_2, \ldots a_{2n}, t$.

It follows that

$$\sum_{i=1}^{i=n}\left(p_i\frac{dq_i}{da_r}\right)=\int_{t_1}^{t}\frac{d\mathrm{L}}{da_r}dt+\mathrm{A}_r$$

$\frac{d\mathrm{L}}{da_r}$ being expressed in terms of $a_1, a_2, \ldots a_{2n}, t$, and integrated with regard to t explicitly ; A_r is a function of $a_1, a_2, \ldots a_{2n}$, and does not contain t explicitly ; t_1 may be any function of $a_1, a_2, \ldots a_{2n}$, but must not contain t.

Now

$$\frac{d}{da_r}\int_{t_1}^{t}\mathrm{L}dt=\int_{t_1}^{t}\frac{d\mathrm{L}}{da_r}dt-(\mathrm{L})_{t=t_1}\frac{dt_1}{da_r};$$

if we put

$$\mathrm{A}_r+(\mathrm{L})_{t=t_1}\frac{dt_1}{da_r}+\mathrm{B}_r=0$$

we have

and similarly,

$$\left.\begin{aligned}\sum_{i=1}^{i=n}\left(p_i\frac{dq_i}{da_r}\right)&=\frac{d}{da_r}\int_{t_1}^{t}\mathrm{L}dt-\mathrm{B}_r\\ \sum_{i=1}^{i=n}\left(p_i\frac{dq_i}{da_s}\right)&=\frac{d}{da_s}\int_{t_1}^{t}\mathrm{L}dt-\mathrm{B}_s\end{aligned}\right\}\quad\ldots\ldots\ (2)$$

If we differentiate the first of these expressions with respect to a_s and the second with respect to a_r, on the supposition that all the functions involved are expressed in terms of $a_1, a_2, \ldots a_{2n}, t$, we find that

$$[a_r, a_s]=\frac{d\mathrm{B}_s}{da_r}-\frac{d\mathrm{B}_r}{da_s}\quad\ldots\ldots\ldots\ldots\ (3)$$

where

$$[a_r, a_s]=\sum_{i=1}^{i=n}\left(\frac{dq_i}{da_r}\frac{dp_i}{da_s}-\frac{dq_i}{da_s}\frac{dp_i}{da_r}\right)\quad\ldots\ldots\ldots\ (4)$$

Since B_r, B_s are independent of the time explicitly, it is clear that $[a_r, a_s]$ does not contain t explicitly when it is expressed in terms of $a_1, a_2, \ldots a_{2n}$. It follows also that we can assign any value we please to t in equation (2) as far as regards the determination of B_r.

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If

$$C_r = B_r + \frac{dA}{d\alpha_r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

where A is any arbitrary function of $\alpha_1, \alpha_2, \ldots \alpha_{2n}$ it is clear that

$$\left[\alpha_r, \alpha_s\right] = \frac{dC_s}{d\alpha_r} - \frac{dC_r}{d\alpha_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

If we put $t = t_1$ in equation (2)

$$C = -\left[\sum_{i=1}^{i=n}\left(p_i \frac{dq}{d\alpha_r}\right) + L\frac{dt_1}{d\alpha_r}\right]_{t=t_1} + \frac{dA}{d\alpha_r} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

Having thus determined $[\alpha_r, \alpha_s]$, we can now proceed in the usual manner to find the new equations.

Let the original equations be

$$\dot{q}_i = \frac{d(H+\Omega)}{dp_i}, \quad \dot{p}_i = -\frac{d(H+\Omega)}{dq_i} \; (i = 1, 2, \ldots n) \quad . \quad . \quad . \qquad (8)$$

Then, since $\alpha_1, \alpha_2, \ldots \alpha_{2n}$ have been so chosen that when $q_1, q_2, \ldots q_n, p_1, p_2, \ldots p_n$ are expressed in terms of them and t,

$$\frac{dq_i}{dt} = \frac{dH}{dp_i}, \quad -\frac{dp_i}{dt} = \frac{dH}{dq_i} \; (i = 1, 2, \ldots n)$$

it follows that

$$\sum_{r=1}^{r=2n} \frac{dq_i}{d\alpha_r}\dot{\alpha}_r = \frac{d\Omega}{dp_i}, \quad -\sum_{r=1}^{r=2n} \frac{dp_i}{d\alpha_r}\dot{\alpha}_r = \frac{d\Omega}{dq_i} \; (i = 1, 2, \ldots n).$$

Multiplying each of these equations by

$$\frac{dp_i}{d\alpha_s}, \; \frac{dq_i}{d\alpha}$$

respectively, and adding the complete set, we see that

$$\frac{d\Omega}{d\alpha_s} = \sum^{r=2n} [\alpha_r, \alpha_s]\dot{\alpha}_r$$

$$= \sum_{r=1}^{r=2n}\left(\frac{dC_s}{d\alpha_r} - \frac{dC_r}{d\alpha_s}\right)\dot{\alpha}_r$$

$$\therefore \; \frac{d\Omega}{d\alpha_s} = \dot{C}_s - \sum_{r=1}^{r=2n} \frac{dC_r}{d\alpha_s}\dot{\alpha}_r (s = 1, 2, \ldots n) \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

are the required equations.

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§ 2. *Application to the Planetary Theory.*

The motion of the planets is governed in general principally by the attractive force of the Sun, which would, if acting alone, cause each planet to describe an ellipse. The deviations from the elliptic orbit are small; it is an advantage to deal with variables whose rates of variation are small, for it is then possible to use methods of approximation. In some expositions of the planetary theory the first change of variables is to the constants of the elliptic orbit. The set which are usually denoted by a, e, ϵ, g, α, ϕ, are uncanonical.

Let the origin of coordinates be taken at the Sun's centre, which is supposed to be at rest or moving with uniform velocity in a straight line. If (x, y, z) are the coordinates of the planet referred to three fixed rectangular axes through the origin

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{r}$$

when the units are properly chosen.
Then

$$p_1 = \dot{x}, \; p_2 = \dot{y}, \; p_3 = \dot{z}$$

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) - \frac{1}{r}.$$

Let b denote any one of the constants a, e, ϵ, g, α, ϕ. It is necessary to find

$$\sum_i p_i \frac{dq_i}{db} = \dot{x}\frac{dx}{db} + \dot{y}\frac{dy}{db} + \dot{z}\frac{dz}{dz}.$$

We shall now consider the transformation to new rectangular axes—

$$\left.\begin{aligned} x &= l_1\xi + l_2\eta + l_3\zeta \\ y &= m_1\xi + m_2\eta + m_3\zeta \\ z &= n_1\xi + n_2\eta + n_3\zeta \end{aligned}\right\}$$

where the direction cosines may be functions of the elements, but not of t explicitly.

$$\begin{aligned} \dot{x}\frac{dx}{db} + \dot{y}\frac{dy}{db} + \dot{z}\frac{dz}{db} = {} & (l_1\dot{\xi} + l_2\dot{\eta} + l_3\dot{\zeta})\left(l_1\frac{d\xi}{db} + l_2\frac{d\eta}{db} + l_3\frac{d\zeta}{db} + \xi\frac{dl_1}{db} + \eta\frac{dl_2}{db} + \zeta\frac{dl}{db}\right) \\ & + (m_1\dot{\xi} + m_2\dot{\eta} + m_3\dot{\zeta})\left(m_1\frac{d\xi}{db} + m_2\frac{d\eta}{db} + m_3\frac{d\zeta}{db} + \xi\frac{dm_1}{db} + \eta\frac{dm_2}{db} + \zeta\frac{dm_3}{db}\right) \\ & + (n_1\dot{\xi} + n_2\dot{\eta} + n_3\dot{\zeta})\left(n_1\frac{d\xi}{db} + n_2\frac{d\eta}{db} + n_3\frac{d\zeta}{db} + \xi\frac{dn_1}{db} + \eta\frac{dn_2}{db} + \zeta\frac{dn_3}{db}\right) \\ = {} & \dot{\xi}\frac{d\xi}{db} + \dot{\eta}\frac{d\eta}{db} + \dot{\zeta}\frac{d\zeta}{db} \\ & + (\dot{\xi}\eta - \eta\dot{\xi})\left(l_2\frac{dl_1}{db} + m_2\frac{dm_1}{db} + n_2\frac{dn_1}{db}\right) \\ & + (\eta\dot{\zeta} - \zeta\dot{\eta})\left(l_3\frac{dl_2}{db} + m_3\frac{dm_2}{db} + n_3\frac{dn_2}{db}\right) + (\zeta\dot{\xi} - \xi\dot{\zeta})\left(l_1\frac{dl_3}{db} + m_1\frac{dm_3}{db} + n_1\frac{dn_3}{db}\right). \end{aligned}$$

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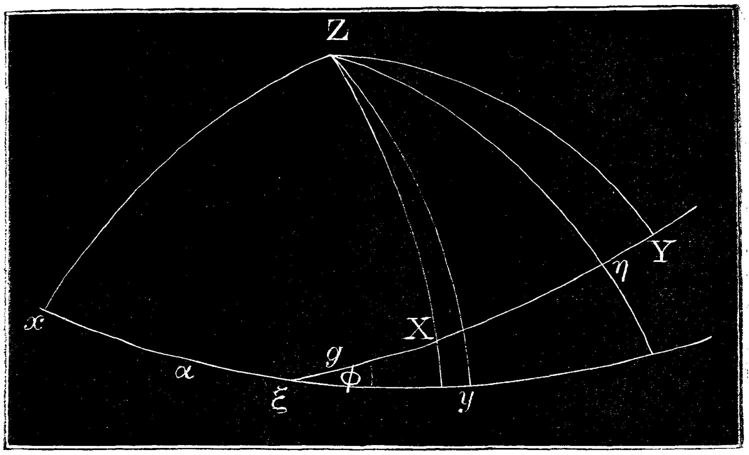
Let the plane $x\,y$ cut the plane of the ecliptic in the axis ξ, and let the axis η be in the plane of the ecliptic. The angle between x and ξ



is α, and if X denote the axis which is taken along the major axis of the ellipse, the angle between ξ and X is g; the inclination of the ecliptic to the plane $x\,y$ is ϕ; a and e have their usual meanings in an ellipse, and ϵ is the epoch.

The solution of the elliptic motion is given by the following equations—

$$\left.\begin{aligned} \mathrm{X} &= a\,(\cos u - e) \\ \mathrm{Y} &= a\sqrt{1-e^2}\,.\,\sin u \\ a^{-\frac{3}{2}}t + \epsilon &= u - e\sin u \end{aligned}\right\}$$

$$\left.\begin{aligned} \xi &= \mathrm{X}\cos g - \mathrm{Y}\sin g \\ \eta &= \mathrm{X}\sin g + \mathrm{Y}\cos g \end{aligned}\right\}$$

$$\left.\begin{aligned} x &= \xi\cos\alpha - \eta\sin\alpha\cos\phi \\ y &= \xi\sin\alpha + \eta\cos\alpha\cos\phi \\ z &= \qquad\quad \eta\sin\phi \end{aligned}\right\}$$

Considering the description of area we see that

$$\xi\dot{\eta} - \eta\dot{\xi} = \mathrm{X}\dot{\mathrm{Y}} - \mathrm{Y}\dot{\mathrm{X}} = \sqrt{a(1-e^2)}$$

Hence

$$\dot{x}\frac{dx}{d\phi} + \dot{y}\frac{dy}{d\phi} + \dot{z}\frac{dz}{d\phi} = 0$$

$$\dot{x}\frac{dx}{d\alpha} + \dot{y}\frac{dy}{d\alpha} + \dot{z}\frac{dz}{d\alpha} = \sqrt{a(1-e^2)}\,.\,\cos\phi$$

$$\begin{aligned} \dot{x}\frac{dx}{dg} + \dot{y}\frac{dy}{dg} + \dot{z}\frac{dz}{dg} &= \dot{\xi}\frac{d\xi}{d\xi} + \dot{\eta}\frac{d\eta}{dg} = \mathrm{X}\dot{\mathrm{Y}} - \mathrm{Y}\dot{\mathrm{X}} \\ &= \sqrt{a(1-e^2)} \end{aligned}$$

and when $b = a$, e or ϵ

$$\dot{x}\frac{dx}{db} + \dot{y}\frac{dy}{db} + \dot{z}\frac{dz}{db} = \dot{\mathrm{X}}\frac{d\mathrm{X}}{db} + \dot{\mathrm{Y}}\frac{d\mathrm{Y}}{db}.$$

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Also

$$L = H + \frac{2}{r} = \frac{2}{r} - \frac{1}{2a}.$$

We put

$$t_1 = -\epsilon a^{\frac{3}{2}}, \text{ so that } u = 0 \text{ when } t = t_1$$

Then

$$L_{t=t_1} = \frac{2}{a(1-e)} - \frac{1}{2a}.$$

Hence using the formula

$$C_b = -\left[\sum_{i=1}^{i=n} p_i \frac{dq_i}{db} - L\frac{dt_1}{db}\right]_{t=t_1} + \frac{dA}{db}$$

we see that, if we assume A to be independent of g, a, ϕ,

$$C_g = -\sqrt{a(1-e^2)}\,;\ C_\alpha = -\sqrt{a(1-e^2)}\,.\cos\phi\,;\ C_\phi = 0.$$

We have still to find C_a, C_e, C_ϵ.

Since

$$\left.\begin{aligned} X &= a(\cos u - e), \\ Y &= a\sqrt{1-e^2}\sin u, \\ a^{-\frac{3}{2}}t + \epsilon &= u - e\sin u \end{aligned}\right\}$$

$$(X)_{u=0} = a(1-e), \quad (Y)_{u=0} = 0$$

$$\left(\frac{dX}{du}\right)_{u=0} = 0, \quad \left(\frac{dY}{du}\right)_{u=0} = a\sqrt{1-e^2}$$

$$(\dot{X})_{u=0} = 0, \text{ and}$$

$$\left(\dot{Y}\frac{dY}{db}\right)_{u=0} = \left[\left(\frac{dY}{du}\right)^2 \frac{du}{dt}\cdot\frac{du}{db}\right]_{u=0} = \sqrt{a}\,(1+e)\left(\frac{du}{db}\right)_{u=0}$$

Hence

$$C_b = -\left(\frac{2}{a(1-e)} - \frac{1}{2a}\right)\frac{dt_1}{db} - \sqrt{a}\,(1+e)\left(\frac{du}{db}\right)_{\substack{u=0\\t=t_1}} + \frac{dA}{db}$$

$$C_a = \frac{3}{2}\epsilon\frac{1}{\sqrt{a}}\left(\frac{2}{1-e} - \frac{1}{2}\right) - \frac{3}{2}\frac{\epsilon}{\sqrt{a}}\frac{1+e}{1-e} + \frac{dA}{da} = \frac{3}{4}\frac{\epsilon}{\sqrt{a}} + \frac{dA}{da}.$$

$$C_e = \frac{dA}{de}$$

$$C_\epsilon = \sqrt{a}\left(\frac{2}{1-e} - \frac{1}{2}\right) - \sqrt{a}\frac{1+e}{1-e} + \frac{dA}{d\epsilon} = \frac{\sqrt{a}}{2} + \frac{dA}{d\epsilon}.$$

If we put $A = -\frac{1}{2}\sqrt{a}\,.\,\epsilon$, we see that

$$C_a = \frac{1}{2}\frac{\epsilon}{\sqrt{a}}, \quad C_e = 0, \quad C_\epsilon = 0,$$

$$C_g = -\sqrt{a(1-e^2)}, \quad C_\alpha = -\sqrt{a(1-e^2)}\cos\phi, \quad C_\phi = 0.$$

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Hence from equation (9), the new equations are

$$\left.\begin{aligned}
\frac{d\Omega}{da} &= \frac{1}{2}a^{-\frac{1}{2}}\dot{\epsilon} + \frac{1}{2}a^{-\frac{1}{2}}\sqrt{1-e^2}(\dot{g} + \dot{\alpha}\cos\phi) \\
\frac{d\Omega}{de} &= \qquad -a^{\frac{1}{2}}\frac{e}{\sqrt{1-e^2}}(\dot{g} + \dot{\alpha}\cos\phi) \\
\frac{d\Omega}{d\epsilon} &= \frac{1}{2}a^{-\frac{1}{2}}\dot{a} \\
\frac{d\Omega}{dg} &= -\frac{1}{2}a^{-\frac{1}{2}}\sqrt{1-e^2}\,.\,\dot{a} + a^{\frac{1}{2}}\frac{e}{\sqrt{1-e^2}}\,.\,\dot{e} \\
\frac{d\Omega}{da} &= -\frac{1}{2}a^{-\frac{1}{2}}\sqrt{1-e^2}\,.\,\dot{a}\cos\phi + a^{\frac{1}{2}}\frac{e}{\sqrt{1-e^2}}\dot{e}\cos\phi + \sqrt{a(1-e^2)}\sin\phi \\
\frac{d\Omega}{d\phi} &= -\sqrt{a(1-e^2)}\,.\,\sin\phi\,.\,\dot{\alpha}.
\end{aligned}\right\}$$

§ 3. *A Method of Successive Approximation.*

In proceeding to consider consecutive terms of the disturbing function, it is assumed that a solution containing $2n$ arbitrary constants $\beta_1, \beta_2, \ldots \beta_{2n}$ has been found of the equations

$$\frac{d\Omega_1}{d\alpha_s} = \dot{C}_s - \sum_{r=1}^{=2n}\frac{dC_r}{d\alpha_s}\dot{\alpha}_r,\ (s = 1, 2, \ldots 2n) \quad \ldots\ldots \quad (10)$$

and it is required to find the equations for $\dot{\beta}_1, \dot{\beta}_2, \ldots \dot{\beta}_{2n}$ when Ω_2 is added to Ω_1.

Let

$$L_1 = \Omega_1 + \sum_{r=1}^{r=2n} C_r\,\dot{\alpha}_r \quad \ldots\ldots\ldots \quad (11)$$

$$\begin{aligned}
\frac{dL_1}{d\beta_k} &= \sum_{s=1}^{s=2n}\frac{d\Omega_1}{d\alpha_s}\frac{d\alpha_s}{d\beta_k} + \sum_{r=1}^{r=2n}\sum_{s=1}^{s=2n}\frac{dC_r}{d\alpha_s}\dot{\alpha}_r\frac{d\alpha_s}{d\beta_k} + \sum_{r=1}^{r=2n} C_r\frac{d\dot{\alpha}_1}{d\beta_k} \\
&= \sum_{s=1}^{s=2n}\left[\dot{C}_s\frac{d\alpha_s}{d\beta_k} + C_s\frac{d\dot{\alpha}_s}{d\beta_k}\right], \text{ by (10)} \\
&= \frac{d}{dt}\left(\sum_{s=1}^{s=2n} C_s\frac{d\alpha}{d\beta_k}\right)
\end{aligned}$$

the terms in the bracket being expressed in terms of $\beta_1, \beta_2, \ldots \beta_{2n}$, t, and $\dot{C}_s$, $\dot{\alpha}_s$ being written in the above for the values of

$$\frac{dC_s}{dt}, \frac{d\alpha_s}{dt}$$

when C_s, α_s are expressed in terms of $\beta_1, \beta_2, \ldots \beta_{2n}$, t.

It follows that

$$\sum_{s=1}^{s=2n} C_s\frac{d\alpha_s}{d\beta_k} = \int_{t_1}^{t}\frac{dL_1}{d\beta_k}dt + D_k,$$

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$\frac{dL_1}{d\beta_k}$ being expressed in terms of $\beta_1, \beta_2, \ldots \beta_{2n}, t$, and integrated with regard to t explicitly ; D_k is a function of $\beta_1, \beta_2, \ldots \beta_{2n}$, but does not contain t explicitly ; t_1 may be any function of $\beta_1, \beta_2, \ldots \beta_{2n}$, but must not contain t.

If

$$D_k + (L_1)_{t=t_1} \frac{dt_1}{d\beta_k} - E_k = 0$$

then

$$\left.\begin{aligned} &\sum_{s=1}^{s=2n}\left(C_s \frac{d\alpha_s}{d\beta_k}\right) = \frac{d}{d\beta_k}\int_{t_1}^{t} L_1\, dt - E_k \\ \text{similarly}\quad &\sum_{s=1}^{s=2n}\left(C_s \frac{d\alpha_s}{d\beta_j}\right) = \frac{d}{d\beta_j}\int_{t_1}^{t} L_1\, dt - E_j \end{aligned}\right\} \quad \ldots\ldots \quad (12)$$

Hence

$$[\beta_k, \beta_j] = \frac{dE_j}{d\beta_k} - \frac{dE_k}{d\beta_j} \quad \ldots\ldots\ldots \quad (13)$$

where

$$[\beta_k, \beta_j] \equiv \sum_{s=1}^{s=2n}\left(\frac{dC_s}{d\beta_j}\,\frac{d\alpha_s}{d\beta_k} - \frac{dC_s}{d\beta_k}\cdot\frac{d\alpha_s}{d\beta_j}\right) \quad \ldots\ldots \quad (14)$$

Since E_j, E_k are independent of t explicitly, it follows that $[\beta_k, \beta_j]$ does not contain t explicitly, when expressed in terms of $\beta_1, \beta_2, \ldots \beta_{2n}$. It follows, also, that, as far as regards the determination of E_k, we can assign any value we please to t in equation (12).

If

$$F_k = E_k + \frac{dB}{d\beta_k} \quad \ldots\ldots \quad (15)$$

when B is any arbitrary function of $\beta_1, \beta_2, \ldots \beta_{2n}$, it is clear that

$$[\beta_k, \beta_j] = \frac{dF_j}{d\beta_k} - \frac{dF_k}{d\beta_j} \quad \ldots\ldots \quad (16)$$

and if we put $t=t_1$ in equation (12)

$$F_k = -\left[\sum_{s=1}^{s=2n}\left(C_s \frac{d\alpha_s}{d\beta_k}\right) + L_1 \frac{dt_1}{d\beta_k}\right]_{t=t} + \frac{dB}{d\beta_k} \quad \ldots \quad (17)$$

Having thus determined $[\beta_k, \beta_j]$ we can now proceed to find the new equations.

Let the original equations be

$$\frac{d(\Omega_1 + \Omega_2)}{d\alpha_s} = \dot{C}_s - \sum_{r=1}^{r=2n}\frac{dC_r}{d\alpha_s}\,\dot{\alpha}_r, \; (s = 1, \ldots 2n) \quad \ldots \quad (18)$$

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Then, since $\beta_1, \beta_2, \ldots \beta_{2n}$ have been so chosen that when $\alpha_1, \alpha_2, \ldots \alpha_{2n}$, $C_1, C_2, \ldots C_{2n}$, are expressed in terms of them and t

$$\frac{d\,\Omega_1}{d\alpha_s} = \frac{dC_s}{dt} - \sum_{r=1}^{r=2n} \frac{dC_r}{d\,\alpha_s} \cdot \frac{d\alpha_r}{dt}, \; (s = 1, 2, \ldots 2n)$$

it follows that

$$\frac{d\,\Omega_2}{d\alpha_s} = \sum_{k=1}^{k=2n} \frac{dC_s}{d\beta_k} \dot{\beta}_k - \sum_{k=1}^{k=2n} \sum_{r=1}^{r=2n} \frac{dC_r}{d\,\alpha_s} \cdot \frac{d\alpha_r}{d\beta_k} \cdot \dot{\beta}_k.$$

Multiplying this equation by

$$\frac{d\alpha_s}{d\beta_j} \; (s = 1, 2, \ldots 2n)$$

and adding the complete set we see that

$$\begin{aligned} \frac{d\Omega_2}{d\beta_j} &= \sum_{k=1}^{k=2n} \sum_{s=1}^{s=2n} \frac{dC_s}{d\beta_k} \cdot \frac{d\alpha_s}{d\beta_j} \dot{\beta}_k \\ &\quad - \sum_{k=1}^{k=2n} \sum_{r=1}^{r=2n} \sum_{s=1}^{s=2n} \frac{dC_r}{d\alpha_s} \cdot \frac{d\alpha_s}{d\beta_j} \cdot \frac{d\alpha_r}{d\beta_k} \cdot \dot{\beta}_k \\ &= \sum_{k=1}^{k=2n} \sum_{s=1}^{s=2n} \frac{dC_s}{d\beta_k} \frac{d\alpha_s}{d\beta_j} \dot{\beta}_k \\ &\quad - \sum_{k=1}^{k=2n} \sum_{r=1}^{r=2n} \frac{dC_r}{d\beta_j} \frac{d\alpha_r}{d\beta_k} \dot{\beta}_k \\ &= \sum_{k=1}^{k=2n} [\beta_k, \beta_j] \, \dot{\beta}_k \\ &= \sum_{k=1}^{k=2n} \left(\frac{dF_j}{d\beta_k} - \frac{dF_k}{d\,\beta_j} \right) \dot{\beta}_k \end{aligned}$$

$$\therefore \frac{d\Omega_2}{d\,\beta_j} = \dot{F}_j - \sum_{k=1}^{k=2n} \frac{dF_k}{d\,\beta_j} \dot{\beta}_k, \; (j = 1, 2, \ldots 2n) \quad . \quad . \quad . \quad . \quad (19)$$

are the required equations.

It is clear that that this theorem includes that of §1 as a particular case. If it is possible to express $\beta_1, \beta_2, \ldots \beta_{2n}$ in terms of $F_1, F_2, \ldots F_{2n}$, we see, by multiplying equation (19) by

$$\frac{d\beta_j}{dF_m}$$

and adding, that

$$-\frac{d\Omega_2}{dF_m} = \dot{\beta}_m - \sum_{k=1}^{k=2n} \frac{d\beta_k}{dF_m} \cdot \dot{F}_k, \; (m = 1, 2 \ldots 2n) \quad . \quad . \quad . \quad . \quad (20)$$

There is a species of reciprocity between the set of variables $(F_1, F_2, \ldots F_{2n})$ and the set $(\beta_1, \beta_2, \ldots \beta_{2n})$.

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From a consideration of the coefficient of $\dot{F}_s$ in (20), it is clear that

$$[F_r, F_s] = -\frac{d\beta_s}{dF_r} + \frac{d\beta_r}{dF_s}.$$

§ 4. If n of the functions denoted by C, say $C_{n+1}, C_{n+2}, \ldots C_{2n}$, are functions of the n α's only, which do not correspond to them, viz. $\alpha_1, \alpha_2, \ldots \alpha_n$, it is possible to indicate a method of solution of the equations :

$$\frac{d\Omega}{d\alpha_s} = \dot{C}_s - \sum_{r=1}^{r=2n} \frac{dC_r}{d\alpha_s} . \ \dot{\alpha}_r . \ (s = 1, 2, \ldots 2n) \quad . \quad . \quad . \quad . \quad (9)$$

which is similar to Jacobi's method of solution of the canonical equations.

Let $\beta_1, \beta_2, \ldots \beta_n, \gamma_1, \gamma_2, \ldots \gamma_n$, be $2n$ variables connected with $\alpha_1, \alpha_2, \ldots \alpha_{2n}$ by the equations

$$\left.\begin{aligned} \frac{d\psi}{d\alpha_i} &= C_i - \sum_{r=n+1}^{r=2n} \alpha_r \frac{dC_r}{d\alpha_i} \\ \frac{d\psi}{d\beta_i} &= \gamma_i \end{aligned}\right\} (i = 1, 2, \ldots n) \quad . \quad . \quad . \quad . \quad (21)$$

where ψ is a function of $\alpha_1, \alpha_2, \ldots \alpha_n, \beta_1, \beta_2, \ldots \beta_n, t$ only.

Let δ denote an arbitrary variation of the variables.

Then since

$$\delta C_r = \sum_{i=1}^{i=n} \frac{dC_r}{d\alpha_i} \delta\alpha_i + \frac{dC_r}{dt} \delta t$$

if C_r do not contain $\alpha_{n+1}, \alpha_{n+2}, \ldots \alpha_{2n}$, where $r = n + 1, n + 2, \ldots 2n$.

$$\begin{aligned} \delta\psi &= \sum_{i=1}^{i=n} \left(\frac{d\psi}{d\alpha_i} \delta\alpha_i + \frac{d\psi}{d\beta_i} \delta\beta_i \right) + \frac{d\psi}{dt} \delta t \\ &= \sum_{i=1}^{i=n} (C_i \, \delta\alpha_i + \gamma_i \delta\beta_i) - \sum_{i=n+1}^{i=2n} \alpha_i \delta \, C_i + \phi\delta t \end{aligned}$$

when

$$\phi \equiv \frac{d\psi}{dt} + \sum_{i=n+1}^{i=2n} \alpha_i \frac{dC}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Let Δ denote another independent arbitrary variation.

Then

$$\begin{aligned} \Delta\delta\psi = &\sum_{i=1}^{i=n} [\Delta C_i \delta\alpha_i + \Delta\gamma_i \delta\beta_i + C_i \Delta\delta\alpha_i + \gamma_i \Delta\delta\beta_t] \\ &- \sum_{i=n+1}^{i=2n} (\Delta\alpha_i \delta C_i + \alpha_i \Delta\delta C_i) + \Delta\phi \, . \, \delta t + \phi \, . \, \Delta\delta t \end{aligned}$$

Similarly

$$\begin{aligned} \delta\Delta\psi = &\sum_{i=1}^{i=n} [\delta C_i \Delta\alpha_i + \delta\gamma_i \Delta\beta_i + C_i \delta\Delta\alpha_i + \gamma_i \delta\Delta\beta_i] \\ &- \sum_{i=n+1}^{i=2n} [\delta\alpha_i \Delta C_i + \alpha_i \delta\Delta C_i] + \delta\phi \, . \, \Delta t + \phi \, . \, \delta\Delta t. \end{aligned}$$

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But

$$\Delta\delta\psi = \delta\Delta\psi,\ \delta\Delta\alpha_i = \Delta\delta\alpha_i, \text{ \&c.}$$

Hence

$$\sum_{i=1}^{i=2n}[\delta\alpha_i\Delta C_i - \Delta\alpha_i\delta C_i] + \sum_{r=1}^{r=n}[-\delta\gamma_r\Delta\beta_r + \Delta\gamma_r\delta\beta_r] + \Delta\phi\,.\,\delta t - \delta\phi\,.\,\Delta t = 0\,. \quad (23)$$

Now $\alpha_1, \alpha_2, \ldots \alpha_{2n}$ can be expressed in terms of $\beta_1, \beta_2, \ldots \beta_n, \gamma_1, \gamma_2, \ldots \gamma_n, t$, and *vice versâ*.

Let $(\quad)'$ denote that the quantity enclosed in the bracket is expressed in terms of $\beta_1, \beta_2, \ldots \beta_n, \gamma_1, \gamma_2, \gamma_n, t$; and let all quantities except ψ, which are not enclosed in $(\quad)'$, be in future expressed in terms of $\alpha_1, \alpha_2, \ldots \alpha_{2n}, t$; but let ψ be in all cases expressed in terms of $\alpha_1, \alpha_2, \ldots \alpha_n, \beta_1\ \beta_2, \ldots \beta_n, t$.

Then

$$\Delta\alpha_i = \sum_{r=1}^{r=n}\left[\frac{d(\alpha_i)'}{d\beta_r}\Delta\beta_r + \frac{d(\alpha_i)'}{d\gamma_r}\Delta\gamma_r\right] + \frac{d(\alpha_i)'}{dt}\Delta t$$

$$\delta\beta_r = \sum_{i=1}^{i=2n}\left[\frac{d\beta_r}{d\alpha_i}\,\delta\alpha_i\right] + \frac{d\beta_r}{dt}\delta t, \text{ \&c.}$$

Equation (23) can then be written

$$\begin{aligned}
&\sum_{i=1}^{i=2n}\left[\delta\alpha_i\left\{\sum_{r=1}^{r=n}\left(\frac{d(C_i)'}{d\beta_r}\Delta\beta_r + \frac{d(C_i)'}{d\gamma_r}\Delta\gamma_r\right) + \frac{d(C_i)'}{dt}\Delta t\right\}\right.\\
&-\sum_{s=1}^{s=2n}\left[\sum_{r=1}^{r=n}\left\{\frac{d(\alpha_s)'}{d\beta_r}\Delta\beta_r + \frac{d(\alpha_s)'}{\delta\gamma_r}\Delta\gamma_r\right\} + \frac{d(\alpha_s)'}{dt}\Delta t\right].\left[\sum_{i=1}^{i=n}\frac{dC_s}{d\alpha_i}\delta\alpha_i + \frac{dC_s}{dt}\delta t\right]\\
&+\sum_{r=1}^{r=n}\left[\Delta\gamma_r\left\{\sum_{i=1}^{i=2n}\frac{d\beta_r}{d\alpha_i}\delta\alpha_i + \frac{d\beta_r}{dt}\delta t\right\}\right.\\
&\left.-\Delta\beta_r\left\{\sum_{i=1}^{i=2n}\frac{d\gamma_r}{d\alpha_i}\delta\alpha_i + \frac{d\gamma_r}{dt}\delta t\right\}\right]\\
&+\delta t\left\{\sum_{r=1}^{r=n}\left(\frac{d(\phi)'}{d\beta_r}\Delta\beta_r + \frac{d(\phi)'}{d\gamma_r}\Delta\gamma\right) + \frac{d(\phi)'}{dt}\Delta t\right\}\\
&-\Delta t\left\{\sum_{i=1}^{i=2n}\frac{d\phi}{d\alpha_i}\delta\alpha_i + \frac{d\phi}{dt}\delta t\right\} = 0 \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (24)
\end{aligned}$$

But the variations

$$\delta\alpha_1, \delta\alpha_2, \ldots \delta\alpha_{2n}, \delta t, \Delta\beta_1, \Delta\beta_2, \ldots \Delta\beta_n, \Delta\gamma_1, \Delta\gamma_2, \ldots \Delta\gamma_n, \Delta t$$

are all arbitrary and independent; each coefficient of these quantities in equation (24) must therefore be zero.

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Hence we find :

$$\left.\begin{aligned}\frac{d(\mathrm{C}_i)'}{d\beta_r}-\sum_{s=1}^{s=2n}\frac{d(\alpha_s)'}{d\beta_r}\cdot\frac{d\mathrm{C}_s}{d\alpha_i}-\frac{d\gamma_r}{d\alpha_i}=0\\ \frac{d(\mathrm{C}_i)'}{d\gamma_r}-\sum_{s=1}^{s=2n}\frac{d(\alpha_s)'}{d\gamma_r}\cdot\frac{d\mathrm{C}_s}{d\alpha_i}+\frac{d\beta_r}{d\alpha_i}=0\end{aligned}\right\}\quad\ldots\ldots\quad(25)$$

$$\frac{d(\mathrm{C}_i)'}{dt}-\sum_{s=1}^{s=2n}\frac{d(\alpha_s)'}{dt}\cdot\frac{d\mathrm{C}_s}{d\alpha_i}-\frac{d\phi}{d\alpha_i}=0\quad\ldots\ldots\quad(26)$$

$$\left.\begin{aligned}-\frac{d\gamma_r}{dt}-\sum_{s=1}^{s=2n}\frac{d(\alpha_s)'}{d\beta_r}\cdot\frac{d\mathrm{C}_s}{dt}+\frac{d(\phi)'}{d\beta_r}=0\\ \frac{d\beta_r}{\delta t}-\sum_{s=1}^{s=2n}\frac{d(\alpha_s)'}{d\gamma_r}\cdot\frac{d\mathrm{C}_s}{dt}+\frac{d(\phi)'}{d\gamma_r}=0\end{aligned}\right\}\quad\ldots\ldots\quad(27)$$

$$\frac{d(\phi)'}{dt}-\sum_{s=1}^{s=2n}\frac{d(\alpha_s)'}{dt}\cdot\frac{d\mathrm{C}_s}{dt}-\frac{d\phi}{dt}=0\quad\ldots\ldots\quad(28)$$

Now the original equations are

$$\frac{d\Omega}{d\alpha_s}=\dot{\mathrm{C}}_s-\sum_{r=1}^{r=2n}\frac{d\mathrm{C}_r}{d\alpha_s}\dot{\alpha}_r,\ (s=1,\ 2,\ \ldots\ 2n)\quad\ldots\ldots\quad(9)$$

Hence

$$\begin{aligned}\frac{d(\Omega)'}{d\beta_r}&=\sum_{s=1}^{s=2n}\left[\frac{d(\alpha_s)'}{d\beta_r}\cdot\dot{\mathrm{C}}_s-\sum_{i=1}^{i=2n}\frac{d\mathrm{C}_i}{d\alpha_s}\cdot\frac{d(\alpha_s)'}{d\beta_r}\cdot\dot{\alpha}_i\right]\\ &=\sum_{s=1}^{s=2n}\left[\frac{d(\alpha_s)'}{d\beta_r}\cdot\dot{\mathrm{C}}_s\right]-\sum_{i=1}^{i=2n}\dot{\alpha}_i\frac{d(\mathrm{C}_i)'}{d\beta_r}\\ &=\sum_{s=1}^{s=2n}\left[\frac{d(\alpha_s)'}{d\beta_r}\cdot\dot{\mathrm{C}}_s-\frac{d(\mathrm{C}_s)'}{d\beta_r}\cdot\dot{\alpha}_s\right]\\ &=\sum_{s=1}^{s=2n}\left[\frac{d\mathrm{C}_s}{dt}\cdot\frac{d(\alpha_s)'}{d\beta_r}+\sum_{i=1}^{i=2n}\frac{d\mathrm{C}_s}{d\alpha_i}\dot{\alpha}_s\frac{d(\alpha_s)'}{d\beta_r}\right]\\ &\quad-\sum_{i=1}^{i=2n}\left[\dot{\alpha}_i\left\{\sum_{s=1}^{s=2n}\frac{d(\alpha_s)'}{d\beta_r}\cdot\frac{d\mathrm{C}_s}{d\alpha_i}+\frac{d\gamma_r}{d\alpha_i}\right\}\right]\text{ by (25)}\\ &=-\frac{d\gamma_r}{dt}+\frac{d(\phi)'}{d\beta_r}-\sum_{i=1}^{i=2n}\dot{\alpha}_i\frac{d\gamma_r}{d\alpha_i},\text{ by (27)}\end{aligned}$$

whence we obtain

Similarly

$$\left.\begin{aligned}\dot{\gamma}_r&=-\frac{d}{d\beta_r}(\phi-\Omega)'\\ \dot{\beta}_r&=\frac{d}{d\gamma_r}(\phi-\Omega)'\end{aligned}\right\}\ (r=1,\ 2,\ \ldots\ n).\quad.\quad(29)$$

These new equations are canonical.

If $\phi=\Omega$, we see that $\beta_1,\ \beta_2,\ \ldots\ \beta_n,\ \gamma_1,\ \gamma_2,\ \ldots\ \gamma_n$, form a

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canonical set of arbitrary constants, the solution being given by the equations

$$\left.\begin{aligned}\frac{d\psi}{d\alpha_i} &= C_i - \sum_{r=n+1}^{r=2n} \alpha_r \frac{dC_r}{d\alpha_i}\\ \frac{d\psi}{d\beta_i} &= \gamma_i\end{aligned}\right\} (i=1, 2, \dots n) \quad . \; . \; . \; . \quad (21)$$

when

$$\phi = \frac{d\psi}{dt} + \sum_{i=n+1}^{i=2n} \alpha_i \frac{dC_i}{dt} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (22)$$

From the equations

$$\frac{d\psi}{d\alpha_i} = C_i - \sum_{r=n+1}^{r=2n} \alpha_r \frac{dC_r}{d\alpha_i}, \quad (i=1, 2, \dots n)$$

we can in general find $\alpha_{n+1}, \alpha_{n+2}, \dots \alpha_{2n}$ in terms of

$$\alpha_1, \alpha_2, \dots \alpha_n, \frac{d\psi}{d\alpha_1}, \frac{d\psi}{d\alpha_2}, \dots \frac{d\psi}{d\alpha_n}, t.$$

If these values are substituted in the equation

$$\frac{d\psi}{dt} + \sum_{i=1}^{i=n} \alpha_i \frac{dC_i}{dt} = \Omega \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (30)$$

we obtain a differential equation of the first order, for ψ in terms of $\alpha_1, \alpha_2, \dots \alpha_n, t$. If any complete solution of this equation, containing n arbitrary constant $\beta_1, \beta_2, \dots \beta_n$ be found, the solution of the equations (9) is given by equations (21).

If, now, a further term Ω_1 is added to Ω, the equations for the variations of the arbitrary constants are (by [29]) seen to be

$$\left.\begin{aligned}\dot{\gamma}_r &= \frac{d\Omega_1}{d\beta_r}\\ -\dot{\beta}_r &= \frac{d\Omega_1}{d\gamma_r}\end{aligned}\right\} (r=1, 2, \dots n) \quad . \; . \; . \; . \; . \; . \; . \quad (31)$$

which form a canonical set.

In particular, if $C_{n+1}, C_{n+2}, \dots C_{2n}$, are all zero or definite constants, we see, by putting

$$\psi = \sum_{i=1}^{i=n} \alpha_i \beta_i$$

in which $\phi = 0$, that $\alpha_1, \alpha_2, \dots \alpha_n, C_1, C_2, \dots C_n$, in this case form a canonical set of variables. For example, in the case of the planetary theory,

$$\begin{array}{ccc} a, & g, & \alpha \\ \frac{1}{2}\frac{\epsilon}{\sqrt{a}}, & -\sqrt{a(1-e^2)}, & -\sqrt{a(1-e^2)} \, . \cos \phi \end{array}$$

form a canonical set of arbitrary constants of the elliptic motion.

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Note on Mr. A. Y. G. Campbell's paper "On the Variation of Uncanonical Arbitrary Constants, with an Application to the Planetary Theory." By Sir Robert S. Ball, LL.D., F.R.S.

The object of this note is to set down another method by which the calculation of Lagrange's "parentheses" in the planetary theory can be effected very simply. The method was suggested by a manuscript which Mr. Campbell kindly communicated to me. I have also to acknowledge the assistance derived from the instructive remarks of Mr. Cowell and Mr. Whittaker, to whom, as well as to Mr. Campbell, it has been my privilege to have lectured at Cambridge.

Adopting the notation of Mr. Campbell's manuscript, we put

$$\frac{\partial(x, \dot{x})}{\partial(\lambda, \mu)} \text{ for } \frac{dx}{d\lambda} \cdot \frac{d\dot{x}}{d\mu} - \frac{dx}{d\mu} \cdot \frac{d\dot{x}}{d\lambda},$$

where λ, μ are any two of the six arbitrary constants.

We thus have the expression for the "parenthesis"

$$[\lambda, \mu] = \frac{\partial(x, \dot{x})}{\partial(\lambda, \mu)} + \frac{\partial(y, \dot{y})}{\partial(\lambda, \mu)} + \frac{\partial(z, \dot{z})}{\partial(\lambda, \mu)}.$$

We now make the transformation to another set of rectangular coordinates x_1, y_1, z_1, where

$$\begin{aligned} x &= l_1 x_1 + l_2 y_1 + l_3 z_1 \\ y &= m_1 x_1 + m_2 y_1 + m_3 z_1 \\ z &= n_1 x_1 + n_2 y_1 + n_3 z_1 \end{aligned}$$

It is unnecessary to write down the general result. For the planetary theory we take the special case where

$$\begin{array}{lll} l = \cos\alpha & l_2 = -\sin\alpha\cos\phi & l_3 = 0 \\ m_1 = \sin\alpha & m_2 = \cos\alpha\cos\phi & m_3 = 0 \\ n_1 = 0 & n_2 = \sin\phi & n_3 = 0 \end{array}$$

And if we express x_1 by ξ and y_1 by η we get after a little reduction

$$[\lambda, \mu] = \frac{\partial(\xi, \dot{\xi})}{\partial(\lambda, \mu)} + \frac{\partial(\eta, \dot{\eta})}{\partial(\lambda, \mu)} - \sin\phi\,(\xi\dot{\eta} - \dot{\xi}\eta)\frac{\partial(\alpha, \phi)}{\partial(\lambda, \mu)}$$

$$+ \cos\phi\,\frac{d\alpha}{d\lambda}\,\frac{d}{d\mu}(\xi\dot{\eta} - \dot{\xi}\eta) - \cos\phi\,\frac{d\alpha}{d\mu}\,\frac{d}{d\lambda}(\xi\dot{\eta} - \dot{\xi}\eta)$$

We now transform to two new variables X and Y given by the equations

$$\xi = X\cos g - Y\sin g$$

$$\eta = X\sin g + Y\cos g$$